tory convergence in 25 iterations for N up to 10 stages with initial guesses far removed from the correct values. The values obtained agree with those given by Frank (*11*) for $N = 2,3,5,8$, and 10 stages with $A_0 = B_{N-1} = 1.0$, $A_{N-1} = B_0 = 0$, $k_1\theta = 4$, and initial guesses $A_n^{(0)} = B_n^{(0)} = 0.5$ for $n = 1, \ldots, N$. This method is much more stable than the iterative methods discussed by Frank. Also it is much simpler and faster than the technique finally adopted by Frank of solving the transient differential equations of the system numerically in order to obtain the steady state values. The advantages relative to this last method can be expected to increase with the complexity of the reaction system.

This example clearly indicates that the development of efficient computational schemes for process design often depends on the direct examination of the equations describing individual process units. The next paper of this series presents a systematic means of representing and utilizing the structural properties of the system of equations associated with optimizing a process design.

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The Estimation of Parameters for a Commonly Used Stochastic Model

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A region into which particles arrive in a random manner, remain a random amount of time, and then leave is considered. This model is used in penetration theories of heat and mass transfer. From observations of the number of particles present at any time, it is desired to estimate arrival and exit statistics, residence time statistics, and average rates of transfer across the region. Assuming arrival is a Poisson process, equations governing the above statistics are derived. Some problems in spectral analysis arising from the use of nondifferentiable stochastic processes are solved. Estimators for important parameters are discussed, and it is shown that generally they are biased. A derivation linking the rate of transfer across the region with the rates of transfer of particles is obtained and compared with other such results.

Consider a region into which particles arrive in a random manner, remain a random amount of time, and then leave. This paper is concerned with the particular problem of estimating various physical properties of interest exclusively from counts of the number of particles present in the region at any time. This problem appeared in the literature in 1916 (*14*) in connection with investigations of colloid statistics, and again in 1953 (*1*) in connection with estimating the speed of organisms. In 1954, under the assumption that the arrival process is Poisson, general results concerning properties of the number of particles in the region were derived (*2*). Recently, in application to heat transfer in fluidized beds (*3*), a set of assumptions about the physical nature of the arrival process led to results which, in part, were identical to those obtained (*2*). Because the previous work has been done in widely differing fields and on different aspects of this problem, and also because of a number of unusual features connected with the estimation problem, the authors felt further discussion of the problem to be worthwhile. In particular, this paper discusses the estimation of the following quantities: entrance and exit statistics of the particles, residence times of the particles, and transfer properties of the region, in order to show that consistent estimators of the above properties are biased. The use of both covariances and spectra are discussed.

THEORY

Let $R(t)$ be a random variable which represents the number of particles that enter a region in a time interval $(0, t)$, and assume that $R(t)$ is a Poisson process with parameter λ, so that

$$p_R(i) = P[R(t) = i] = \frac{e^{-\lambda t}(\lambda t)^i}{i!}, \quad i = 0, 1, 2 \ldots \quad (1)$$

Let $N(t)$ be a random variable which represents the number of particles in the region at time t (henceforth called the population), and let T_j be a non-negative continuous random variable which represents the residence time of the jth particle in the region. Assume that the T_j's are independent of each other and of $R(t)$, and identically distributed with mean θ and finite variance. Let $F_T(x)$ and $f_T(x)$ be the distribution function and density function, respectively, of T_j. Let A_j be a random variable which represents the time of arrival of the jth particle into the region. Use will be made of the following theorem (4).

Let $R(t)$ be a Poisson process with parameter λ. Under the condition that k events have occurred in the interval $(0, t)$ the k times $t_1, t_2, \ldots, t_k$ at which events occur, considered as unordered random variables, are distributed uniformly and independently in the interval $(0, t)$.

Thus, in what follows, the A_j will be taken as uniformly distributed in $(0, t)$. Henceforth the subscript on A and T will be dropped.

The probability mass function of the population at time t is (5)

$$p_N(j) = P[N(t) = j]$$

$$= \sum_{i=0}^{\infty} P[R(t) = j + i]\ P$$

$$[i \text{ particles leave the region before } t \mid R(t) = j + i]$$

$$= \sum_{i=0}^{\infty} p_R(j+i) \binom{j+i}{i} a_t^i (1-a_t)^k$$

$$= \frac{e^{-\lambda t(1-a_t)}[\lambda t(1-a_t)]^j}{j!}, \quad j = 0, 1, 2, \ldots, \qquad (2)$$

where

$$a_t = P\,[\text{a particle arrives and leaves in } (0, t)]$$

$$= P[A + T \leq t] = \frac{1}{t}\int_0^t (t-x) f_T(x)\ dx \qquad (3)$$

Estimators will later be discussed only for the stationary process, and so it is noted that because T has finite mean θ (6),

$$\lim_{t\to\infty} \lambda t(1-a_t) = \lambda\theta = \overline{N} \qquad (4)$$

and

$$\lim_{t\to\infty} p_N(j) = \frac{e^{-\overline{N}}\overline{N}^j}{j!}, \quad j = 0, 1, 2, \ldots \qquad (5)$$

The joint moment generating function of the population is

$$\psi(\alpha, \beta) = E\ e^{\alpha N(t) + \beta N(t+\tau)}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} P[N(t) = i]$$

$$\times P\,[N(t+\tau) = j \mid N(t) = i]\ e^{\alpha i + \beta j} \qquad (6)$$

Following (3), let

$$N(t+\tau) = L(\tau) + S(t, \tau) \qquad (7)$$

where $L(\tau)$ is the number of particles which arrive at the region during $(t, t+\tau)$ and are still present at $t+\tau$, and $S(t, \tau)$ is the number of particles present at t and still present at $t+\tau$. Because of the properties of the arrival Poisson process, $L(\tau)$ is independent of $N(t)$ and of $S(t, \tau)$. Thus

$$P[L(\tau) + S(t, \tau) = j \mid N(t) = i]$$

$$= \sum_{k=\min[0,\, j-i]}^{j} P[N(\tau) = k]\ P$$

$$[S(t+\tau) = j - k \mid N(t) = i]$$

$$= \sum_{k=\min[0,\, j-i]}^{j} e^{-\lambda\tau(1-a_\tau)} \frac{[\lambda\tau(1-a_\tau)]^k}{k!}$$

$$\binom{i}{j-k} q_{t,\tau}^{j-k} (1-q_{t,\tau})^{i-j+k} \qquad (8)$$

where

$$q_{t,\tau} = P\,[\text{a particle leaves after } t+\tau \mid \text{the particle was present at } t]$$

$$= \frac{P[A \leq t,\ A + T \geq t+\tau]}{P[A \leq t,\ A + T \geq t]}$$

$$= \frac{\int_\tau^{t+\tau} (\tau - x) f_T(x)\ dx + t\int_{t+\tau}^\infty f_T(x)\ dx}{\int_0^t x f_T(x)\ dx + t\int_t^\infty f_T(x)\ dx} \qquad (9)$$

and a_τ is given by Equation (3) with t replaced by τ.

As before, only stationary results will be of interest, and so it is noted that

$$q_\tau = \lim_{t\to\infty} q_{t,\tau} = \frac{1}{\theta}\int_\tau^\infty (\tau - x) f_T(x)\ dx \qquad (10)$$

and

$$\lambda\tau(1-a_\tau) = \overline{N}(1-q_\tau) \qquad (11)$$

By using Equations (2), (8), (10), and (11) in Equation (6)

$$\overline{\psi}(\alpha, \beta) = \lim_{t\to\infty} \psi(\alpha, \beta) = e^{-\overline{N}[2 - q_\tau(1 - e^\alpha - e^\beta + e^{\alpha+\beta}) - e^\alpha - e^\beta]} \qquad (12)$$

The stationary covariance of the population, first derived in (2), may now be computed as

$$\Gamma(\tau) = \lim_{t\to\infty}\{E\ N(t)\,N(t+\tau) - E\ N(t)\ E\ N(t+\tau)\}$$

$$= \frac{\partial^2\overline{\psi}}{\partial\alpha\,\partial\beta}\Bigg|_{\substack{\alpha=0\\ \beta=0}} - \overline{N}^2 = \lambda \int_\tau^\infty (x-\tau) f_T(x)\ dx$$

$$= \overline{N} - \lambda\int_\tau^\infty P[T > x]\ dx \qquad (13)$$

for $\tau \geq 0$, with

$$\Gamma(-\tau) = \Gamma(\tau) \qquad (13a)$$

In what follows only the case $\tau \geq 0$ will be considered, relations of Equation (13*a*) being implied for negative τ.

Note that

$$\Gamma'(\tau) = -\lambda\int_\tau^\infty f_T(x)\ dx \qquad (14)$$

where the prime denotes differentiation with respect to τ, so that

$$\Gamma'(0_+) = -\lambda \qquad [\text{observe that } \Gamma'(0_-) = \lambda] \qquad (15)$$

and

$$\Gamma''(\tau) = \lambda f_T(\tau) \qquad (16)$$

Note that $\Gamma'(0)$ is zero for stationary stochastic processes which are (mean square) differentiable (7), but as $N(t)$ is not a differentiable process, Equation (15) does not present a paradox. Recalling that for an exponential $f_T(x)$ the probability of staying an additional length of time t is independent of the time already spent in residence, it is seen that under such a condition $N(t)$ is a Markov immigration-death process (8); otherwise it may not be a Markov process.

In dealing with stationary processes, it is often convenient to deal with the spectral density, rather than the covariance. The assumption that T has a finite variance leads to the fact that the stationary covariance (13) is then absolutely integrable from $(-\infty, \infty)$; it thus has a power spectral density function $p_N(\omega)$ defined by the equation

$$p_N(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}\Gamma(\tau)\,d\tau \qquad (17)$$

Let

$$\begin{aligned} g(\omega) &= \int_0^\infty \Gamma(\tau)\, e^{-i\omega\tau}\,d\tau \\ &= \lambda \int_0^\infty f_T(x) \int_0^x (x-\tau)\, e^{-i\omega\tau}\,d\tau dx \\ &= \lambda\left[\frac{ET}{i\omega} + \frac{1}{(i\omega)^2}\varphi_T(-\omega) - \frac{1}{(i\omega)^2}\right] \end{aligned} \qquad (18)$$

where the characteristic function of T is defined as

$$\varphi_T(\omega) = \int_0^\infty f_T(x)\, e^{i\omega x}\,dx$$

Then, from Equations (17) and (18),

$$\begin{aligned} p_N(\omega) &= \frac{1}{2\pi}[g(\omega) + g(-\omega)] = \\ &= \frac{\lambda}{2\pi\omega^2}[2 - \varphi_T(\omega) - \varphi_T(-\omega)] \\ &= \frac{\lambda}{\pi\omega^2}\int_0^\infty (1-\cos\omega x)\, f_T(x)dx \end{aligned} \qquad (19)$$

a result first obtained in (2). This formula is of interest in the following; therefore, it will be examined in some detail.

Note that since $f_T(x)$ is zero for $x < 0$, $\varphi_T(\omega)$ is complex with the properties

$$\begin{cases} \varphi_T(0) = 1 \\ \varphi_T(\omega) \to 0 \quad \text{as} \quad \omega \to \infty \\ \varphi_T(-\omega) = \text{complex conjugate of } \varphi_T(\omega) \end{cases} \qquad (20)$$

It is clear that $p_N(\omega)$ is real and symmetric in ω. Next, note that

$$p_N(\omega) \simeq \frac{\lambda}{\pi\omega^2} \text{ for large } \omega \qquad (21)$$

and that

$$p_N(0) = \frac{\lambda}{2\pi} E\,T^2 \qquad (22)$$

Thus, $p_N(\omega)$ is finite at the origin, decreases like $\lambda/\pi\omega^2$ for large ω, and is absolutely integrable from $-\infty$ to ∞. While the formula

$$\Gamma(\tau) = \int_{-\infty}^{\infty} p_N(\omega)\, e^{i\omega\tau}\,d\omega \qquad (23)$$

is valid, differentiation of it with respect to τ to obtain a formula for $\Gamma'(\tau)$ is not justified since

$$\int_{-\infty}^{\infty} |\omega|\; p_N(\omega)\,d\omega$$

is unbounded. Moreover, since $N(t)$ is not differentiable in mean square and since $\Gamma'(\tau)$ may be interpreted as the covariance of $N(t)\,\dot{N}(t+\tau)$, where t is large and the dot over $N(t+\tau)$ denotes time differentiation, the usual interpretation of $\Gamma'(\tau)$ is also not valid.

To obtain a formula for $\Gamma'(\tau)$ in terms of $p_N(\omega)$, note first that $\Gamma'(\tau)$ has a Fourier transform given by

$$\begin{aligned} p_{\Gamma'}(\omega) &= -\frac{\lambda}{\pi}\int_0^\infty \left(\frac{\sin\omega x}{\omega}\right) f_T(x)\,dx \\ &= \frac{-\lambda}{2\pi i\,\omega}[\varphi_T(\omega) - \varphi_T(-\omega)] \end{aligned} \qquad (24)$$

A study of Equations (19) and (24) shows that $p_N(\omega)$ is a function of just the real part of $\varphi_T(\omega)$, while $p_{\Gamma'}(\omega)$ is a function of just the imaginary part of $\varphi_T(\omega)$. Thus,

$$p_{\Gamma'}(\omega) = \left[\omega\; p_N(\omega) - \frac{\lambda}{\pi\omega}\right]\gamma \qquad (25)$$

where

$$\gamma = \tan^{-1}\frac{\int_0^\infty f_T(x)\sin\omega x\,dx}{\int_0^\infty f_T(x)\cos\omega x\,dx} \qquad (26)$$

Since in (26), $p_{\Gamma'}(\omega)$ is related to $p_N(\omega)$ through $f_T(x)$, this is not a useful result. In order to eliminate $\varphi_T(\omega)$ from the relationship between $p_{\Gamma'}(\omega)$ and $p_N(\omega)$, the Hilbert transform H, which relates the real and imaginary parts of a complex function (9), can be used. It can be seen that

$$\begin{aligned} p_{\Gamma'}(\omega) &= \frac{-\lambda}{\pi\omega} H\left[1 - \frac{\pi x^2}{\lambda}p_N(x)\right] \\ &= \frac{1}{\omega\pi}\,\mathcal{P}\int_{-\infty}^{\infty}\frac{x^2 p_N(x)\,dx}{\omega - x} \end{aligned} \qquad (27)$$

where $\mathcal{P}$ denotes the principal value of the integral, and thus

$$\Gamma'(\tau) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega\tau}}{\omega}\,\mathcal{P}\int_{-\infty}^{\infty}\frac{x^2 p_N(x)}{\omega - x}\,dx\,d\omega \qquad (28)$$

However, and this is an interesting point, it can be shown that the formula one would use if $N(t)$ were differentiable, namely

$$\Gamma'(\tau) = \int_{-\infty}^{\infty} i\,\omega e^{i\omega\tau}\,p_N(\omega)\,d\omega \qquad (29)$$

is correct except at $\tau = 0$. By substituting Equation (19) into Equation (29), the result is Equation (14) if $\tau \neq 0$.

In order to relate $\Gamma''(\tau)$ to $p_N(\omega)$, it is seen that the Fourier transform of $\Gamma''(\tau)$ is

$$\begin{aligned} p_{\Gamma''}(\omega) &= \frac{\lambda}{\pi}\int_0^\infty f_T(x)\cos\omega x\,dx \\ &= \frac{\lambda}{2\pi}[\varphi_T(\omega) + \varphi_T(-\omega)] \end{aligned} \qquad (30)$$

and eliminating $\varphi_T(\omega)$ from Equations (19) and (30) yields

$$p_{\Gamma''}(\omega) = \frac{\lambda}{\pi} - \omega^2\; p_N(\omega) \qquad (31)$$

and thus

$$\Gamma''(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau}\left[\frac{\lambda}{\pi} - \omega^2 p_N(\omega)\right]d\omega \qquad (32)$$

for all τ including $\tau = 0$.

Note that in the case of $\Gamma''(\tau)$, another differentiation of Equation (29) does not yield the correct result.

Consideration of the number of particles which leave the region in the interval $(0, t)$, denoted by the random variable $D(t)$, leads to

$$p_D(k) = P[D(t) = k]$$

$$= \sum_{i=0}^{\infty} P[R(t) = k + i]\, P[D(t) = k | R(t) = k + i]$$

$$= \sum_{i=0}^{\infty} \frac{(\lambda t)^{k+i} e^{-\lambda t}}{(k+i)!} \binom{k+i}{k} a_t^k (1-a_t)^i$$

$$= \frac{(\lambda t a_t)^k e^{-\lambda t a_t}}{k!}, \quad k = 0, 1, 2 \ldots \quad (33)$$

As t becomes large, it is seen that

$$p_D(k) \simeq \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k = 0, 1, 2, \ldots \quad (34)$$

In order to investigate the fluctuations in the population, consider the random variable

$$\Delta N(\tau) = \lim_{t \to \infty} |N(t+\tau) - N(t)| \quad (35)$$

Then for $j > 0$,

$$p_{\Delta N}(j)$$
$$= P[\Delta N(\tau) = j] = 2 \lim_{t\to\infty} P[N(t+\tau) - N(t) = j]$$
$$= 2 \lim_{t\to\infty} P[H(\tau) - V(t,\tau) = j] \quad (36)$$

and

$$p_{\Delta N}(0) = \lim_{t\to\infty} P[H(\tau) - V(t,\tau) = 0] \quad (37)$$

where $V(t, \tau)$ is the number of particles present at t that are not present at $t + \tau$. Using the fact that H and V are independent, and Equation (11), it is seen that

$$p_{\Delta N}(j) = \lim_{t\to\infty} 2 \sum_{\substack{l=0 \\ l=m-j}}^{\infty} \sum_{m=0}^{\infty} P[H(\tau) = l] \sum_{i=m}^{\infty} p_N(i) \binom{i}{m} (1-q_{t,\tau})^m\, q_{t,\tau}{}^{i-m}$$

$$= 2\, e^{-2\overline{N}(1-q_\tau)} I_j[2\,\overline{N}\,(1-q_\tau)], \quad j > 0 \quad (38)$$

and similarly,

$$p_{\Delta N}(j) = e^{-2\overline{N}(1-q_\tau)} I_o\,[2\,\overline{N}\,(1-q_\tau)], \quad j = 0 \quad (39)$$

where $I_j(x)$ is a Bessel function of order j with complex argument. Note that the assumption of symmetry in Equation (36) actually follows from carrying out the summations in Equation (38).

Equations similar in form to (38) and (39) were derived (3) by assuming $R(t)$ and $D(t)$ to be independent random variables. This is approximately true only for small τ, and thus the results in (3) are approximations to Equations (38) and (39), and agreement up to terms of order τ in a Taylor's series exists.

TRANSFER PROPERTIES

In order to consider the transfer properties of the region, assume that each particle can transfer a positive quantity h according to the equation

$$h(G) = \int_0^G u(x)\, dx \quad (40)$$

where G is a random variable that denotes the amount of time which the particle has spent in the region if the region is examined at t (sometimes denoted by age). The function $u(t)$ is a time rate of transfer for the particle, and is assumed to be finite for all t. Then the total amount of quantity transferred during $(0, t)$ is (assuming no particles to be present at 0)

$$Q(t) = \sum_{i=0}^{R(t)} h_i(G) \quad (41)$$

Assuming each particle to be independent of the others and of $R(t)$, it can be shown that (10)

$$E\ Q(t) = E\ R(t)\ E\ h(G) = \lambda\ t\ E\ h(G) \quad (42)$$

Now,

$$E\ h(G) = \int_0^t h(g)\, f_G(g)\, dg \quad (43)$$

and given that a particle has arrived during $(0, t)$,

$$F_G(g) = P\,[A + T > t,\ A > t-g] + P\,[A + T \leq t,\ T \leq g]$$

$$= \begin{cases} 0 & , \ g < 0 \\ \dfrac{g}{t}\displaystyle\int_g^\infty f_T(x)\, dx + \int_0^g f_T(x)\, dx, & 0 \leq g < t \\ 1 & , \ t \leq g \end{cases}$$

Thus,

$$f_G(g) = F'_G(g) = \begin{cases} \dfrac{1}{t}\displaystyle\int_g^\infty f_T(x)\, dx + \left(1 - \frac{g}{t}\right) f_T(g), & 0 \leq g < t \\ 0 & , \text{ otherwise} \end{cases} \quad (44)$$

and using Equations (43) and (44) in Equation (42) yields

$$E\ Q(t) = \lambda \int_0^t h(g)\, P\,[T > g]\, dg + \lambda\ t \int_0^t h(g)\, f_T(g)\, dg - \lambda \int_0^t g\ h(g)\, f_T(g)\, dg \quad (45)$$

By using Equation (45), and replacing t by $t' + t$, it can be shown that for the stationary process $(t' \to \infty)$,

$$\frac{d\,E\,Q(t)}{d\ t} = \lambda \int_0^\infty h(g)\, f_T(g)\, dg$$

$$= \lambda \int_0^\infty u(g)\, P\,[T > g]\, dg \quad (46)$$

This result does not depend on the fact that no particles were assumed to be present at 0, as the contribution due to the number of particles initially present goes to zero in the limiting stationary process.

Equation (46) was obtained in (3) on the basis of an assumption which will now be discussed.

It can be shown that for the particular model under discussion (employing a commonly used notation)

$$\frac{P\,[T > g]\, dg}{\theta} =$$

P [a particle is present between g and $g + dg$ | the particle was present at g] (47)

and the quantity on the left hand side of Equation (47) has been equated to an age distribution of elements by Zwieterling (11). With the interpretation given by Equation (47), it is seen that the integral on the right hand side of Equation (46) represents the rate of transfer averaged over particles present in the region. The assumption that the average rate of transfer for the region is equal to the rates of transfer for individual particles averaged over all particles present has been implicitly assumed in the work of Danckwerts (12) and by other authors since then (3). However, because $Q(t)$ is not a differentiable function of time, the integral in Equation (46) does not represent the average rate of transfer of the region, but the rate of average transfer. Because $Q(t)$ is not differentiable, it is only in the above sense that this particulate model under discussion yields a rate of transfer for the region.

ESTIMATORS

Consider sampling the stationary population at intervals of length Δt. Let

$$N_i = N(t + i\Delta t), \quad i = 1, 2, \ldots, n \tag{48}$$

and

$$\Gamma_j = \lim_{t\to\infty} \{E\ N(t + i\Delta t)\ N(t + i\Delta t + j\Delta t) - E\ N(t + i\Delta t)\quad E\ N(t + i\Delta t + j\Delta t)\} = \overline{N}q_j, \quad j = 0, 1, \ldots, n-1 \tag{49}$$

An estimator for the average population size, $\overline{N}$, may be found from

$$\hat{N} = \frac{\sum_{i=1}^{n} N_i}{n} \tag{50}$$

where n is the number of observations of $N(t)$.

It is seen that $\hat{N}$ is an unbiased estimator of $\overline{N}$, since

$$E\,\hat{N} = \overline{N} \tag{51}$$

To show that $\hat{N}$ is a consistent estimator of $\overline{N}$, note that the variance of $\hat{N}$ is

$$\text{var}\ \hat{N} = \frac{\overline{N}}{n} + \frac{2}{n}\sum_{j=1}^{n-1}\Gamma_j - \frac{2}{n^2}\sum_{j=1}^{n-1} j\ \Gamma_j \tag{52}$$

and that as n approaches infinity, the use of Equation (13) and $\theta < \infty$ are sufficient for var $\hat{N}$ to approach zero.

In order to determine the statistics of the arrival and departure processes from observations of $N(t)$, examination of Equations (1) and (34) reveal that only an estimate of λ is necessary, and Equation (15) offers a hint for constructing such an estimator. Let

$$\lambda_1 = \frac{-\sum_{i=1}^{n-1} [(N_i - \hat{N})(N_{i+1} - \hat{N}) - \hat{N}]}{(n-1)\Delta t} \tag{53}$$

Then

$$E\lambda_1 = -\frac{\Gamma_1 - \overline{N}}{\Delta t} + \frac{\overline{N}}{n\Delta t} - \frac{2}{(n-1)\Delta t}\sum_{j=1}^{n-1}\Gamma_j - \frac{2(n+1)}{n^2(n-1)}\sum_{j=1}^{n-1} j\ \Gamma_j \tag{54}$$

and

$$\lim_{n\to\infty} E\lambda_1 = \frac{\lambda}{\Delta t}\int_o^{\Delta t} P[T > x]\, dx \tag{55}$$

For small Δt, it is seen that

$$\lim_{n\to\infty} E\lambda_1 \simeq \lambda - \frac{\lambda}{2}\Delta t\ f_T(0) \tag{56}$$

and thus even asymptotically, λ_1 is a biased estimator of λ. For comparison, the estimator used in (3) was

$$\lambda_2 = \lim_{t\to\infty} \frac{\sum_{i=1}^{n-1} |N_{i+1} - N_i|}{2\,\Delta t(n-1)} \tag{57}$$

and using Equation (38), it can be shown that

$$E\,\lambda_2 = \frac{\overline{N}(1 - q_{\Delta t})}{\Delta t} e^{-2\overline{N}(1-q_{\Delta t})} \{I_0[2\overline{N}(1 - q_{\Delta t})] + I_1[2\overline{N}(1 - q_{\Delta t})]\} \simeq \lambda - \lambda\ \Delta t \left[\lambda + \frac{f_T(0)}{2}\right] \tag{58}$$

where the last result holds for small Δt. An estimator suggested by the material in (1) is

$$\lambda_3 = \lim_{t\to\infty} \frac{\sum_{i=1}^{n-1} |N_{i+1} - N_i|^2}{2\,\Delta t(n-1)} \tag{59}$$

and it can be shown that

$$E\,\lambda_3 = \frac{\overline{N} - \Gamma_1}{\Delta t} = \frac{\lambda}{\Delta t}\int_0^{\Delta t} P[T > x]\, dx \tag{60}$$

Thus λ_3 has less bias than both λ_1 and λ_2 for fixed values of n and Δt. All three of these estimators are consistent, but for fixed n, it can be shown that var $\lambda_2 <$ var $\lambda_3 <$ var λ_1, where the last inequality is correct for the practical conditions of $\overline{N}$ large and $\Gamma_1/\overline{N}$ close to one. Equation (22) shows that λ is proportional to a spectral density, and it is well known (13) that unbiased estimates of spectral densities are not consistent. Consistency is obtained only if smoothing is used, and the smoothing procedure introduces a bias. It thus appears that unbiased and consistent estimators of λ are not available. However, there is one situation where it is possible to achieve an unbiased and consistent estimator λ, and that is when a physical situation exists where a particle must spend a minimum time m in the region. Then,

$$P\,[T < m] = 0 \tag{61}$$

and it can be shown that Equation (10) holds for $\tau > m$, and that

$$q_\tau = \frac{1}{\theta}\int_m^\infty (x - \tau)\, f_T(x)\, dx = 1 - \frac{\tau}{\theta}, \quad \tau \leq m \tag{62}$$

Under these conditions, if $\Delta t < m$, λ_3 is an unbiased and consistent estimator of λ, while λ_1 is asymptotically unbiased and consistent.

The estimation of the derivative of the average transfer across the region, given by Equation (46), depends on estimating $\Gamma'(\tau)$, and an estimator of the finite difference of $\Gamma'(\tau)$ at a time $\tau = j\Delta t$ (with $\overline{N}$ assumed known for simplicity) shows that

$$E\,\lambda_4 = E\frac{\sum_{i=1}^{n-j-1} N_i(N_{i+j-1} - N_{i+j})}{(n-j-1)\,\Delta t} = -\frac{\lambda}{\Delta t}\int_{j\Delta t}^{(j+1)\Delta t} P[T > x]\, dx \simeq -\Gamma'(j\Delta t) + \frac{\lambda\Delta t}{2} f_T(j\Delta t) \tag{63}$$

for small Δt. An alternative is to use Equation (28), but the problems inherent in estimating $p_N(\omega)$ will not be discussed in this paper.

No simple estimators could be found for the residence time problem which depended only on the statistics of the population, for λ always appeared coupled with $f_T(x)$. Thus, Equation (4) could be used to estimate θ by taking the ratio of estimates for $\overline{N}$ and λ, and similarly the second moment of T could be estimated using Equation (21). The properties of such estimators were not investigated, for they depend on joint distribution functions of higher order than were derived in this paper.

CONCLUSIONS

A commonly used stochastic model was discussed, and some basic formulas previously scattered in the literature were rederived. New results in terms of the spectral density of the population were obtained, and some interesting problems arising from the nondifferentiability of the random functions involved were discussed. Expressions for the probability mass functions of the exit process and of the absolute value of the fluctuations of the population were derived. This last result was used to compute the bias of various consistent estimators of the arrival rate. A physical situation where one can obtain zero bias was illustrated.

A derivation of a formula previously used in studies of heat transfer across a region was also presented.

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NOTATION

a_τ, a_t = probability of a particle leaving before the subscripted time
A, A_j = random variable representing the time of arrival of a particle
$D(t)$ = random variable representing the number of particles leaving in the interval $(0, t)$
E = expectation operator
$f_T(x), f_G(x)$ = probability density function of the subscripted random variable
$F_T(x), F_G(x)$ = probability distribution function of the subscripted random variable
g = age
$g(\omega)$ = one sided Fourier transform of $\Gamma(\tau)$
G = random variable representing the age of a particle
$h(G)$ = random function representing the amount of quantity transferred
$H[\ ]$ = Hilbert transform
$I_j(x)$ = Bessel function of order j with imaginary argument
$L(\tau)$ = random variable representing the number of particles which arrive during $(t, t+\tau)$ and are still present at $t+\tau$
m = minimum residence time
n = number of observations of $N(t)$
$N(t), N_i$ = random variable representing the number of particles present at times t and $i\Delta t$ respectively
$\overline{N}$ = mean number of particles present in the stationary state
$\hat{N}$ = estimator of $\overline{N}$
$p_D(i)$ = probability mass function of the random variable D
$p_N(j)$ = probability mass function of the random variable N
$p_{\Delta N}(j)$ = probability mass function of the random variable ΔN
$p_R(i)$ = probability mass function of the random variable R
$p_N(\omega)$ = spectral density of $N(t)$ in the stationary state
$p_{\Gamma'}(\omega)$ = Fourier transform of $\Gamma'(\tau)$
$p_{\Gamma''}(\omega)$ = Fourier transform of $\Gamma''(\tau)$
$\mathcal{P}$ = principal value of an integral
$q_{t,\tau}$ = conditional probability that a particle leaves after $t+\tau$, given the particle was present at t
q_τ = the limit of $q_{t,\tau}$ for large t
$Q(t)$ = total amount of quantity transferred in the interval $(0, t)$
$R(t)$ = random variable representing the number of particles that arrive in the interval $(0, t)$
$S(t,\tau)$ = random variable representing the number of particles present at t and still present at $t+\tau$
t = time
T, T_j = random variable representing the residence time of a particle
$u(t)$ = time rate of transfer of a particle
$V(t,\tau)$ = random variable representing the number of particles present at t and not present at $t+\tau$

Greek Letters

α = parameter in joint moment generating function
β = parameter in joint moment generating function
γ = phase angle between components of $\varphi_T(\omega)$
$\Gamma(\tau)$ = stationary covariance of population
Γ_j = $\Gamma(\tau)$ with $\tau = j\Delta t$
$\Delta N(\tau)$ = absolute value of fluctuation of $N(t)$ over an interval of time τ in the stationary state
Δt = sampling interval
θ = mean residence time of a particle
λ = parameter in Poisson probability mass function
$\lambda_1, \lambda_2, \lambda_3$ = estimators of λ
λ_4 = estimator of $\Gamma'(\tau)$
τ = time interval
$\varphi_T(\omega)$ = characteristic function of $f_T(x)$
$\psi(\alpha,\beta)$ = joint moment generating function of $N(t)$ and $N(t+\tau)$
$\overline{\psi}(\alpha,\beta)$ = limiting value of $\psi(\alpha,\beta)$ for large t
ω = frequency, radians/second

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